Exercise 1.7.1

Fib と Fib_{split} は finite products を持つ

Proof

$\begin{matrix} \mathbb{E} \\ \downarrow p \\ \mathbb{B} \end{matrix}$, $\begin{matrix} \mathbb{D} \\ \downarrow q \\ \mathbb{A} \end{matrix}$ を fibration とする

$$p\times q \overset{def}{=} \begin{matrix} \mathbb{E}\times\mathbb{D} \\ \downarrow p\times q \\ \mathbb{B}\times\mathbb{A} \end{matrix}$$

p,q が split のとき $\overline{(u,v)} \overset{def}{=} (\overline{u},\overline{v})$ と定義すれば $p\times q$ も split

---

$\pi : p\times q \to p$

$\pi \triangleq (\pi,\pi)$

$\pi : \mathbb{E}\times\mathbb{D}\to\mathbb{E}$ は明らかに cartesian morphism と splitting を保存

$\pi' : p\times q \to q$ も同様

任意の
$$\begin{tikzcd} \mathbb{E} \arrow[d, "p"] & \mathbb{F} \arrow[l, "H"'] \arrow[r, "G"] \arrow[d, "r"] & \mathbb{D} \arrow[d, "q"'] \\ \mathbb{B} & \mathbb{C} \arrow[l, "K"] \arrow[r, "J"'] & \mathbb{A} \end{tikzcd}$$
に対して、H,G が cartesian morphism を保存するなら $\langle H,G\rangle : \mathbb{F}\to\mathbb{E}\times\mathbb{D}$ も保存

H,G が splitting を保存するなら、$\langle H,G\rangle$ も保存

$\begin{matrix} 1 \\ \downarrow \\ 1 \end{matrix}$ は terminal な split fibration

$$\begin{tikzcd} \mathbb{E} \arrow[r] \arrow[d] & 1 \arrow[d] \\ \mathbb{B} \arrow[r] & 1 \end{tikzcd}$$
は明らかに cartesian morphism と splitting を保存する

Exercise 1.7.2

$\begin{matrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{matrix}$, $\begin{matrix}\mathbb{D}\\ \downarrow q\\ \mathbb{B}\end{matrix}$ are fibration and $H:\mathbb{E}\to\mathbb{D}$ is a functor such that $q\circ H=p$

(i) H is full and faithful

Hf is cartesian $\Rightarrow$ f is cartesian

Proof

$f: X\to Y$ $u=pf$ とする

Hf is cartesian

$\Leftrightarrow$ $\mathbb{D}_v(Z, HX)\xrightarrow{Hf\circ(-)}\mathbb{D}_{u\circ v}(Z, HY)$ is iso for each Z, v

by Exercise 1.1.2

$\Rightarrow$ $\mathbb{D}_v(HW, HX)\xrightarrow{Hf\circ(-)}\mathbb{D}_{u\circ v}(HW, HY)$ is iso for each W, v

$\Leftrightarrow$ $\mathbb{E}_v(W, X)\xrightarrow{f\circ(-)}\mathbb{E}_{u\circ v}(W, Y)$ is iso for each W, v

since H is full and faithful

$\Leftrightarrow$ f is cartesian

by Exercise 1.1.2

(ii) Hが fibred functor とする

- $H:\mathbb{E}\to\mathbb{D}$ is full $\Rightarrow$ every $H_I:\mathbb{E}\to\mathbb{D}$ is full は明らか。
- every $H_I:\mathbb{E}\to\mathbb{D}$ is full $\Rightarrow$ $H:\mathbb{E}\to\mathbb{D}$ is full

Proof $f\in\mathbb{D}(HX, HY)$ とする。$g\in\mathbb{E}(Z, Y)$ を $q(f)$上の cartesian morphism とすると Hg は $q(f)$上の cartesian morphism なので $f=Hg\circ h$ で $h\in\mathbb{E}_{pX}(HX, HZ)$. H_{pX}が full なので $h=Hh'$

よって $f = Hg \circ h = Hg \circ Hh' = H(g \circ h')$

ゆえに H は full

- $H : \mathbb{E} \to \mathbb{D}$ is faithful $\Rightarrow$ every $H_I : \mathbb{E}_I \to \mathbb{D}_I$ is faithful
  は明らか

- every $H_I : \mathbb{E}_I \to \mathbb{D}_I$ is faithful $\Rightarrow$ $H : \mathbb{E} \to \mathbb{D}$ is faithful

Proof

$f, g \in \mathbb{E}(X, Y)$, $Hf = Hg$ とする

$u = qHf = qHg$ とおき $h \in \mathbb{E}(Z, Y)$
を u 上の cartesian morphism
とすると、$f = h \circ f'$, $g = h \circ g'$ で
$Hf = Hh \circ Hf'$, $Hg = Hh \circ Hg'$
Hh は cartesian なので $Hf = Hg$ より
$Hf' = Hg'$
$H_{pX} f' = Hf' = Hg' = H_{pX} g'$ で
H_{pX} が faithful なので $f' = g'$
よって $f = g$

ゆえに H は faithful

Exercise 1.7.3

$2=\{\bot,\top\}$ with $\bot \sqsubseteq \top$

$$
\begin{tikzcd}
\mathrm{Fam}(2) \arrow[rr, "\simeq"] \arrow[dr, "p"'] & & \mathrm{Sub}(\mathrm{Sets}) \arrow[dl, "q"] \\
& \mathrm{Sets} &
\end{tikzcd}
$$

$F : \mathrm{Fam}(2) \to \mathrm{Sub}(\mathrm{Sets})$ を以下のように定義

$$
\begin{tikzcd}
\{X_i\}_{i\in I} \arrow[d, "{(u,\{f_i\}_i)}"] & \longmapsto & \{i\in I \mid X_i=\top\} \arrow[r, hook] \arrow[d, "u"'] & I \arrow[d, "u"] \\
\{Y_j\}_{j\in J} & & \{j\in J \mid Y_j=\top\} \arrow[r, hook] & J
\end{tikzcd}
$$

$f_i : X_i \sqsubseteq Y_{u(i)}$

$G : \mathrm{Sub}(\mathrm{Sets}) \to \mathrm{Fam}(2)$ を以下のように定義

$$
\begin{tikzcd}
& X \arrow[r, tail, "m"] \arrow[d, "f"'] \arrow[dl, mapsto] & I \arrow[d, "u"] \\
\{X_i\}_{i\in I} \arrow[d, "{(u,\{f_i\})}"] & Y \arrow[r, tail, "n"'] & J \\
\{Y_j\}_{j\in J} & &
\end{tikzcd}
$$

$X_i \overset{\mathrm{def}}{=} \top$ if $i\in \mathrm{Im}(m)$

$ \overset{\mathrm{def}}{=} \bot$ otherwise

$Y_j \overset{\mathrm{def}}{=} \top$ if $j\in \mathrm{Im}(n)$

$ \overset{\mathrm{def}}{=} \bot$ otherwise

$f_i : X_i \sqsubseteq Y_{u(i)}$

$X_i=\top \iff i\in \mathrm{Im}(m) \iff \exists x.\ i=m(x) \Rightarrow \exists x.\ u(i)=u(m(x))$

$\Rightarrow \exists x.\ u(i)=n(f(x)) \Rightarrow u(i)\in \mathrm{Im}(n) \iff Y_{u(i)}=\top$

GF=Id は明らか

FG=Id も、$FG(X \overset{m}{\rightarrowtail} I) = (\mathrm{Im}(m) \hookrightarrow I)$ で

$$\begin{tikzcd} X \arrow[r, tail, "m"] \arrow[d, "\cong"'] & I \\ \mathrm{Im}(m) \arrow[ur, hook] & \end{tikzcd}$$

mono として iso なので、

Sub(Sets) で $(X \overset{m}{\rightarrowtail} I) = (\mathrm{Im}(m) \hookrightarrow I)$

Sub(Sets) の object は同値類なので

$$\begin{tikzcd} X \arrow[r, tail, "m"] \arrow[d, "f"'] & I \arrow[d, "u"] \\ Y \arrow[r, tail, "n"'] & J \end{tikzcd}$$

に対して、$FG(u,f)=(u,f)$

$\because$ Sub(Sets)(m,n)
$=$ Sub(Sets)(FGm, FGn)
は高々一要素なので

F が cartesian morphism を保存

$$\{X_{u(i)}\}_i \xrightarrow{(u,\{id\}_i)} \{X_j\}_j \overset{F}{\longmapsto}$$

$$\begin{tikzcd} \{i \in I \mid X_{u(i)} = T\} \arrow[r, "u"] \arrow[d, hook] & \{j \in J \mid X_j = T\} \arrow[d, hook] \\ I \arrow[r, "u"'] & J \end{tikzcd}$$

Exercise 1.7.2 (i) より
G も cartesian morphism
を保存

Exercise 1.7.4

$\mathbb{C} \longmapsto \begin{pmatrix} \mathrm{Fam}(\mathbb{C}) \\ \downarrow \\ \mathrm{Sets} \end{pmatrix}$ が $\mathrm{Cats} \to \mathrm{Fib}_{split}(\mathrm{Sets})$ に拡張できる事を示せ

$F : \mathbb{C} \to \mathbb{D}$ に対して

$$\begin{tikzcd} \mathrm{Fam}(\mathbb{C}) \arrow[rr, "\mathrm{Fam}(F)"] \arrow[dr] & & \mathrm{Fam}(\mathbb{D}) \arrow[dl] \\ & \mathrm{Sets} & \end{tikzcd}$$

を

$$\begin{array}{ccc} \{X_i\}_{i\in I} & \longmapsto & \{FX_i\}_{i\in I} \\ \downarrow (u, \{f_i\}_{i\in I}) & \longmapsto & \downarrow (u, \{Ff_i\}_{i\in I}) \\ \{Y_j\}_{j\in J} & \longmapsto & \{FY_j\}_{j\in J} \end{array}$$

で定義

Cartesian morphism $\{X_{u(i)}\}_{i\in I} \xrightarrow{(u, \{id_{X_{u(i)}}\}_{i\in I})} \{X_j\}_{j\in J}$ は cartesian morphism $\{FX_{u(i)}\}_{i\in I} \longrightarrow \{FX_j\}_{j\in J}$

$(u, \{Fid_{X_{u(i)}}\}_{i\in I}) = (u, \{id_{FX_{u(i)}}\}_{i\in I})$

に移るので、一般のcartesian morphismも保存

Splittingと

Exercise 1.7.5

$I \longmapsto \left(\begin{array}{c}\mathbb{B}/I \\ \downarrow \mathrm{dom}_I \\ \mathbb{B}\end{array}\right)$ を $\mathbb{B} \to \mathrm{Fib}_{split}(\mathbb{B})$ に拡張し、finite product を保存する事を示せ

$I \xrightarrow{u} J$ に対して
$$\begin{tikzcd} \mathbb{B}/I \arrow[rr, "\mathbb{B}/u"] \arrow[dr, "\mathrm{dom}_I"'] & & \mathbb{B}/J \arrow[dl, "\mathrm{dom}_J"] \\ & \mathbb{B} & \end{tikzcd}$$
を

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\varphi"'] & & Y \arrow[dl, "\psi"] \\ & I & \end{tikzcd} \overset{\mathbb{B}/u}{\longmapsto} \begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "u\circ\varphi"'] & & Y \arrow[dl, "u\circ\psi"] \\ & J & \end{tikzcd}$$
で定義

splitting
$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\psi\circ f"'] & & Y \arrow[dl, "\psi"] \\ & I & \end{tikzcd}$$
は splitting
$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "u\circ\psi\circ f"'] & & Y \arrow[dl, "u\circ\psi"] \\ & J & \end{tikzcd}$$
に写されるので $\mathbb{B}/u$ は splitting と cartesian morphism を保存。

$I\times J \longmapsto \left(\begin{array}{c}\mathbb{B}/I\times J \\ \downarrow \mathrm{dom}_{I\times J} \\ \mathbb{B}\end{array}\right)$ で
$$\begin{tikzcd} \mathbb{B}/I\times J \arrow[rr, "\cong"'] \arrow[dr] & & \mathbb{B}/I \times_{\mathbb{B}} \mathbb{B}/J \arrow[dl] \\ & \mathbb{B} & \end{tikzcd}$$
を

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\varphi"'] & & Y \arrow[dl, "\psi"] \\ & I\times J & \end{tikzcd} \longmapsto \left(\begin{tikzcd} X \arrow[rr] \arrow[dr, "\pi\varphi"'] & & Y \arrow[dl, "\pi\psi"] \\ & I & \end{tikzcd} , \begin{tikzcd} X \arrow[rr] \arrow[dr, "\pi'\varphi"'] & & Y \arrow[dl, "\pi'\psi"] \\ & J & \end{tikzcd} \right)$$
で定義

$$1 \longmapsto \begin{pmatrix} \mathbb{B}/1 \\ \downarrow \mathrm{dom}_1 \\ 1 \end{pmatrix}$$ で

$$\begin{tikzcd} \mathbb{B}/1 \arrow[rr, "\cong"] \arrow[dr, "\mathrm{dom}_1"'] & & \mathbb{B} \arrow[dl, "\mathrm{id}"] \\ & \mathbb{B} & \end{tikzcd}$$

を

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & 1 & \end{tikzcd} \longmapsto X \xrightarrow{f} Y$$

で定義

Exercise 1.7.6

let $\mathbb{A}, \mathbb{B}$ be categories with finite product
let $K : \mathbb{A} \to \mathbb{B}$ be a functor presenving these

$$\begin{tikzcd} s(\mathbb{A}) \arrow[r, "s(K)"] \arrow[d] & s(\mathbb{B}) \arrow[d] \\ \mathbb{A} \arrow[r, "K"'] & \mathbb{B} \end{tikzcd}$$

$$(I, X) \xmapsto{s(K)} (KI, KX)$$

$$(u : I \to J,\ f : I \times X \to Y) : (I, X) \to (J, Y)$$
$$\mapsto (Ku,\ Kf \circ r_{I,X}) : (KI, KX) \to (KJ, KY)$$

Exercise 1.3.2より $\mathbb{A}//I$ の finite product は $\mathbb{A}$ と同じ

$s(K) : \mathbb{A}//I \to \mathbb{B}//KI$ が finite product を保存する
ことを示せ

Proof

$$s(K)(1) = s(K)(I, 1) = (KI, K1) \cong (KI, 1) \triangleq 1$$

$$s(K)(X \times Y) = s(K)(I, X \times Y) = (KI, K(X \times Y)) \cong (KI, KX \times KY)$$
$$= KX \times KY = (KI, KX) \times (KI, KY) = s(K)(X) \times s(K)(Y)$$

そういえば $\mathbb{A}//I$ は $I\times -$ comonad の coKleisli category なので、$\mathbb{A}$ の limit が $\mathbb{A}//I$ の limit になって当然

Exercise 1.7.7

Prove the assignment $\begin{pmatrix} \mathbb{A} \\ \downarrow F \\ \mathbb{B} \end{pmatrix} \mapsto \begin{pmatrix} \mathbb{B}\downarrow F \\ \downarrow \\ \mathbb{B} \end{pmatrix}$ yields a functor $\mathbf{Cat}/\mathbb{B} \to \mathbf{Fib}_{split}(\mathbb{B})$, which is left adjoint to inclusion functor.

Proof

$$\begin{tikzcd} \mathbb{A} \arrow[rr, "H"] \arrow[dr, "F"'] & & \mathbb{C} \arrow[dl, "G"] \\ & \mathbb{B} & \end{tikzcd} \quad \overset{L}{\longmapsto} \quad \begin{tikzcd} \mathbb{B}\downarrow F \arrow[rr, "LH"] \arrow[dr, bend right] & & \mathbb{B}\downarrow G \arrow[dl, bend left] \\ & \mathbb{B} & \end{tikzcd}$$ を以下で定義

$LH(I, X, f) = (I, HX, f)$

$LH((h,h') : (I,X,f) \to (J,Y,g)) = (h, Hh')$

$$\begin{tikzcd} I \arrow[r, "h"] \arrow[d, "f"'] & J \arrow[d, "g"] \\ FX \arrow[r, "Fh'"'] & FY \end{tikzcd} \longmapsto \begin{tikzcd} I \arrow[r, "h"] \arrow[d, "f"'] & J \arrow[d, "g"] \\ GHX \arrow[r, "GHh'"'] & GHY \end{tikzcd}$$

$u^*(J,Y,g) = (I, Y, g\circ u)$

$\overline{u}(J,Y,g) : u^*(J,Y,g) \to (J,Y,g)$

$= (u, id_Y)$

$LH(\overline{u}(J,Y,g)) = LH(u, id_Y)$

$= (u, H\, id_Y) = (u, id_{HY})$

$= \overline{u}(J, HY, g)$

より、LHはsplittingを保存

$\eta_F : F \to \mathbb{B}\downarrow F$ in $\mathbf{Cat}/\mathbb{B}$

$\eta_F(X) = (FX, X, id_{FX})$

$\eta_F(h) = (Fh, h)$

$\varepsilon_p : \mathbb{B}\downarrow p \to p$ in $\mathbf{Fib}_{split}(\mathbb{B})$

$\varepsilon_p(I, X, f) = f^*(X)$

$\varepsilon_p(u, h) =$ 右図

$$\begin{tikzcd} I \arrow[r, "u"] \arrow[d, "f"'] & J \arrow[d, "g"] \\ pX \arrow[r, "ph"'] & pY \end{tikzcd} \Rightarrow \begin{tikzcd} & X \arrow[rr, bend left, "h"] & & \\ f^*(X) \arrow[ur] \arrow[r, dashed, "\varepsilon_p(u,h)"'] & g^*(Y) \arrow[rr] & & Y \\ I \arrow[r, "u"'] \arrow[dr, "f"'] & J \arrow[r, "g"'] & pY & \\ & pX \arrow[ur, bend right, "ph"'] & & \end{tikzcd}$$

$\varepsilon_p(\overline{u}(J,Y,g)) = \varepsilon_p(u, id_Y) = \overline{u}(g^*(Y)) = \overline{u}(\varepsilon_p(J,Y,g))$ よりsplittingを保存

$$\varepsilon_p \circ \eta_p(X) = \varepsilon_p(pX, X, id_{pX}) = id^*_{pX}(X) = X$$

$$\varepsilon_p \circ \eta_p(h) = \varepsilon_p(ph, h) = h \quad \text{in } Cat/\mathbb{B}$$

$$\therefore\ \varepsilon_p \circ \eta_p = Id_p$$

$$\begin{tikzcd}
 & X \arrow[rr, bend left, "h"] & & \\
id^*_{pX}(X) \arrow[ur, bend left] \arrow[rr, dashed, "\varepsilon_p(ph,h)"'] & & id^*_{pY}(Y) \arrow[r] & Y
\end{tikzcd}$$

$$pX \xrightarrow{ph} pY \xrightarrow{id_{pY}} pY$$

$$\begin{aligned}
&\varepsilon_{LF} \circ L\eta_F(I, X, f) \\
&= \varepsilon_{LF}(I, \eta_F X, f) \\
&= \varepsilon_{LF}(I, (FX, X, id_{FX}), f) \\
&= f^*(FX, X, id_{FX}) \\
&= (I, X, id_{FX} \circ f) \\
&= (I, X, f)
\end{aligned}$$

$$\boxed{\begin{tikzcd}
\mathbb{B}/F \arrow[d] & f^*(FX, X, id)_{FX} \arrow[r] & (FX, X, f) \\
\mathbb{B} & I \arrow[r, "f"'] & FX
\end{tikzcd}}$$

$$\begin{tikzcd}
 & (FX, X, id_{FX}) \arrow[rr, bend left, "{(Fh, h)}"] & & \\
f^*(FX, X, id_{FX}) \arrow[ur] \arrow[r, dashed] & g^*(FY, Y, id_{FY}) \arrow[rr] & & (FY, Y, id_{FY})
\end{tikzcd}$$

$$I \xrightarrow{u} J \xrightarrow{g} FY$$

$$\begin{aligned}
&\varepsilon_{LF} \circ L\eta_F(u, h) \\
&= \varepsilon_{LF}(u, \eta_F h) \\
&= \varepsilon_{LF}(u, (Fh, h)) \\
&= (u, h)
\end{aligned}$$

$$\therefore\ \varepsilon_{LF} \circ L\eta_F = Id_{LF}$$

よってLは$Fib_{split}(\mathbb{B}) \to Cat/\mathbb{B}$の left adjoint.

Exercise 1.7.8 Verify that $\langle\sigma\rangle$ in Lemma 1.7.10 is a fibred functor $L^*(p) \to K^*(p)$.

$\langle\sigma\rangle : \mathbb{A} \times_L \mathbb{E} \to \mathbb{A} \times_K \mathbb{E}$

$\langle\sigma\rangle(I, X) = (I, \sigma_I^*(X))$

$\langle\sigma\rangle(u, f) = (u, g)$

$$\begin{tikzcd}
\mathbb{A}\times_K\mathbb{E} \arrow[rr, "K'"] \arrow[ddr, bend right, "K^*(p)"'] & & \mathbb{E} \arrow[dd, "p"] \\
& \mathbb{A}\times_L\mathbb{E} \arrow[ul, "\langle\sigma\rangle"] \arrow[ur, "L'"] \arrow[d, "L^*(p)"] \arrow[u, phantom, "\Downarrow\bar{\sigma}"] & \\
& \mathbb{A} \arrow[r, bend left, "K"] \arrow[r, bend right, "L"'] \arrow[r, phantom, "\Downarrow\sigma"] & \mathbb{B}
\end{tikzcd}$$

$$\begin{tikzcd}
& X \arrow[r, "f"] & Y \\
\sigma_I^*(X) \arrow[ur] \arrow[r, dashed, "g"'] & \sigma_J^*(Y) \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
KI \arrow[r, "Ku"'] \arrow[dr, "\sigma_I"'] & KJ \arrow[r, "\sigma_J"'] & LJ \\
& LI \arrow[ur, "Lu"'] &
\end{tikzcd}$$

Proof $\quad u : I \to J,\ f : X \to Y$

(u, f)が $L^*(p)$-cartesianとする

$\overline{\sigma_J} : \sigma_J^*(Y) \to Y$ は p-cartesian

$\overline{\sigma_J} \circ g = f \circ \overline{\sigma_I} : \sigma_I^* \to Y$ は p-cartesian

よって、Exercise 1.1.4 (iii)より g はp-cartesian

ゆえに、$\langle\sigma\rangle(u, f) = (u, g)$ は $K^*(p)$-cartesian

ゆえに$\langle\sigma\rangle$は fibred functor

注: cartesian lifting は up-to-vertical isoで一意なので、(L, L')は cartesian morphismを保存

Exercise 1.7.9

Let (T, η, μ) be a fibred monad on $\begin{matrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{matrix}$.

(i) Show that Kleisli category $\mathbb{E}_T$ is fibred over $\mathbb{B}$.

$$\begin{matrix}\mathbb{E}_T\\ \downarrow p'\\ \mathbb{B}\end{matrix} \qquad \begin{aligned} p'(X') &= p(X) \\ p'(f' : X' \to Y') &= p(f : X \to TY) \\ &= pf : pX \to pTY \\ &= pf : pX \to pY \quad \text{since } T \text{ is fibred} \\ &= pf : p'X' \to p'Y' \end{aligned}$$

$$\begin{aligned} p'(id_{X'}) &= p(\eta_X : X \to TX) \\ &= id_{pX} \quad \text{since } \eta \text{ is vertical} \end{aligned}$$

$$\begin{aligned} p'(g' \circ f') &= p(\mu_Z \circ Tg \circ f) \\ &= p(\mu_Z) \circ p(Tg) \circ p(f) \\ &= p(g) \circ p(f) \quad \text{since } \mu \text{ is vertical and } T \text{ is fibred} \\ &= p'(g') \circ p'(f') \end{aligned}$$

$f : X \to Y$を$u : I \to J$上のcartesian liftingとする
Tはfibred functorなのでTfもu上のcartesian morphism

$$\begin{tikzcd} Z' \arrow[r, dashed, "h'"'] \arrow[rr, bend left, "g'"] & X' \arrow[r, "(\eta_Y \circ f)'"'] & Y' \\ K \arrow[r, "v"'] & I \arrow[r, "u"'] & J \end{tikzcd} \iff \begin{tikzcd} Z \arrow[r, dashed, "h"'] \arrow[rr, bend left, "g"] & TX \arrow[r, "Tf"'] & TY \\ K \arrow[r, "v"'] & I \arrow[r, "u"'] & J \end{tikzcd}$$

$g' = (\eta_Y \circ f)' \circ h'$ $\qquad$ $g = \mu_Y \circ T(\eta_Y \circ f) \circ h = Tf \circ h$

より$(\eta_Y \circ f)'$はuのcartesian lifting

(2) Show that Eilenberg-Moore category $\mathbb{E}^T$ is fibred over $\mathbb{B}$

$$\begin{tikzcd} \mathbb{E}^T \arrow[d, "p'"] \\ \mathbb{B} \end{tikzcd}$$

$p'(X,\varphi) = p(X)$

$p'(h:(X,\varphi)\to(Y,\psi)) = ph$

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, equal] & TX \arrow[d, "\varphi"] \\ & X \end{tikzcd}$$

$$\begin{tikzcd} T^2X \arrow[r, "T\varphi"] \arrow[d, "\mu_X"'] & TX \arrow[d, "\varphi"] \\ TX \arrow[r, "\varphi"'] & X \end{tikzcd}$$

$p\varphi = p\varphi\circ id_{pX}$

$= p\varphi\circ p\eta$

$= p(\varphi\circ\eta) = p\,id_X$

$= id_{pX}$

$f:X\to Y$ を $u:I\to Y$ の cartesian lifting とする

$$\begin{tikzcd} TX \arrow[r, "Tf"] \arrow[d, dashed, "\varphi"'] & TY \arrow[d, "\psi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

で φ を定義

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\eta"'] \arrow[dd, dashed, bend right=60, "id_X"'] & Y \arrow[d, "\eta"] \arrow[dd, bend left=60, "id_Y"] \\ TX \arrow[r, "Tf"'] \arrow[d, "\varphi"'] & TY \arrow[d, "\psi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

id_Y より $\varphi\circ\eta = id$

$$\begin{tikzcd} T^2X \arrow[r, "T^2f"] \arrow[d, "\mu_X"'] & T^2Y \arrow[d, "\mu_Y"] \\ TX \arrow[r, "Tf"'] \arrow[d, "\varphi"'] & TY \arrow[d, "\psi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

かつ

$$\begin{tikzcd} T^2 \arrow[r, "T^2f"] \arrow[d, "T\varphi"'] & T^2Y \arrow[d, "T\psi"] \\ TX \arrow[r] \arrow[d, "\varphi"'] & TY \arrow[d, "\psi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

で $\psi\circ\mu_Y\circ T^2f = \psi\circ T\psi\circ T^2f$ より $\varphi\circ\mu_X = \varphi\circ T\varphi$

$$
\begin{tikzcd}
Z \arrow[r, dashed, "h"] \arrow[rr, bend left, "g"] & X \arrow[r, "f"] & Y \\
K \arrow[r, "v"'] & I \arrow[r, "u"'] & J
\end{tikzcd}
$$

で h を定義

$$
\begin{tikzcd}
(Z,\sigma) \arrow[rr, bend left, "g"] & (X,\varphi) \arrow[r, "f"'] & (Y,\psi) \\
K \arrow[r, "v"'] & I \arrow[r, "u"'] & J
\end{tikzcd}
$$

とすると

$$
\begin{tikzcd}
TZ \arrow[r, dashed] \arrow[rr, bend left, "\psi\circ Tg"] & X \arrow[r, "f"] & Y \\
K \arrow[r, "v"'] & I \arrow[r, "u"'] & J
\end{tikzcd}
$$

$$f\circ(\varphi\circ Th)=\psi\circ Tf\circ Th=\psi\circ Tg$$

$$f\circ(h\circ\sigma)=g\circ\sigma=\psi\circ Tg$$

より $\varphi\circ Th=h\circ\sigma$

よって $f:(X,\varphi)\to(Y,\psi)$ は

u の cartesian lifting