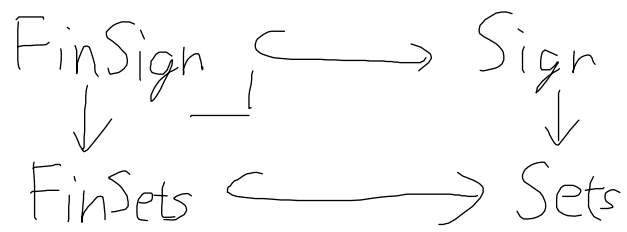


Exercise 1.6.1

group $|\Sigma| = \{G\}$
 $e : \rightarrow G$
 $\cdot : G, G \rightarrow G$
 $(-)^{-1} : G \rightarrow G$

vector space $|\Sigma| = \{K, V\}$
 $\cdot : K, V \rightarrow V$
 $+$: $V, V \rightarrow V$

Exercise 1.6.2



Exercise 1.6.3

$$\begin{array}{ccc} \text{Sign}' \longrightarrow \text{Sets}^{\rightarrow} \\ \downarrow \quad \quad \downarrow \text{cod} \\ \text{Sets} \longrightarrow \text{Sets} \\ T \mapsto T^* \times T \end{array}$$

$$\begin{array}{ccc} \text{Sign} \longrightarrow \text{Fam}(\text{Sets}) \\ \downarrow \quad \quad \downarrow \\ \text{Sets} \longrightarrow \text{Sets} \end{array}$$

(i)

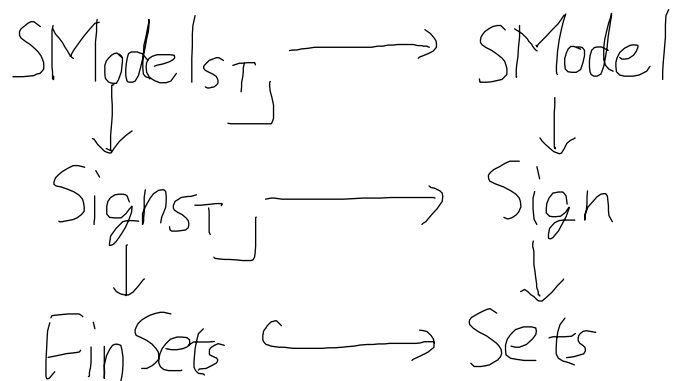
$$\begin{aligned} |\text{Sign}'| &= \{ \langle T, O, f \rangle \mid T, O \in \text{Sets}, f: O \rightarrow T^* \times T \} \\ \text{Sign}'(\langle T, O, f \rangle, \langle T', O', f' \rangle) &= \{ \langle h: T \rightarrow T', h': O \rightarrow O' \rangle \mid (h^* \times h) \circ f \\ &= f' \circ h' \} \end{aligned}$$

$$\begin{array}{ccc} \text{(ii)} \quad \text{Fam}(\text{Sets}) & \xrightarrow{\cong} & \text{Sets}^{\rightarrow} \\ & \searrow \quad \swarrow \text{cod} & \\ & \text{Sets} & \end{array} \quad (\text{Proposition 1.2.2})$$

より明らか

(iii) self evident

Exercise 1.6.4 $SModel_{ST}$ を詳しく



Objects

$(\Sigma, (A_\sigma), \llbracket - \rrbracket)$ which is a S -Model
and $|\Sigma|$ is finite.

Morphisms

S -Model morphism

$$(\phi, (H_\sigma)) : (\Sigma, (A_\sigma), \llbracket - \rrbracket) \rightarrow (\Sigma', (A'_\sigma), \llbracket - \rrbracket')$$

Exercise 1.6.5

Define a category of signatures of predicates
using change-of-base

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\quad} & \text{Fam}(\text{Sets}) \\ \downarrow \quad \lrcorner & & \downarrow \\ \text{Sets} & \xrightarrow{T \mapsto T^*} & \text{Sets} \end{array}$$

Define a category of signatures of both
function and predicate symbols
using change-of-base

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{\quad} & \text{Fam}(\text{Sets}) \times \text{Fam}(\text{Sets}) \\ \downarrow \quad \lrcorner & & \downarrow \\ \text{Sets} & \xrightarrow{\quad} & \text{Sets} \times \text{Sets} \\ & T \mapsto \langle T^* \times T, T^* \rangle & \end{array}$$

Exercise 1.6.6

(i) $X \mapsto \text{Term}(X)$ & $\text{Fam}(\text{Sets})_T \rightarrow S\text{-Model}(\Sigma)$ に拡張せよ

Proof. $f: X \rightarrow Y$ in $\text{Fam}(\text{Sets})_T$ が与えられたとする

$\text{Term}(f) = \text{Term}(X) \rightarrow \text{Term}(Y)$ in $S\text{-Model}(\Sigma)$ と

$\text{Term}(f)(x) = f(x)$

$\text{Term}(f)(F(M_1, \dots, M_n)) = F(\text{Term}(f)(M_1), \dots, \text{Term}(f)(M_n))$

で定義する。

$\text{Term}(id) = id, \text{Term}(f \circ g) = \text{Term}(f) \circ \text{Term}(g)$ (は日月から)

(ii) Assume $((A_\sigma)_{\sigma \in T}, \llbracket - \rrbracket)$ is a Σ -model

and $(p_\sigma: X_\sigma \rightarrow A_\sigma)_{\sigma \in T}$.

$(\llbracket - \rrbracket_p^\sigma: \text{Term}(X)_\sigma \rightarrow A_\sigma)$ が Σ -models の対として

あることを示せ。

Proof

$$\llbracket F(M_1, \dots, M_n) \rrbracket_p = \llbracket F \rrbracket(\llbracket M_1 \rrbracket_p, \dots, \llbracket M_n \rrbracket_p)$$

より

$$\begin{array}{ccc} \text{Term}(X)_{\sigma_1} \times \dots \times \text{Term}(X)_{\sigma_n} & \xrightarrow{\llbracket - \rrbracket_p^{\sigma_1} \times \dots \times \llbracket - \rrbracket_p^{\sigma_n}} & A_{\sigma_1} \times \dots \times A_{\sigma_n} \\ \downarrow F & & \downarrow \llbracket F \rrbracket \\ \text{Term}(X)_\sigma & \xrightarrow{\llbracket - \rrbracket_p^\sigma} & A_\sigma \end{array}$$

が可換なので。

Exercise 1.6.7

Let $(A_\sigma, \models)$ be a Σ -model and ρ a valuation $(X_\sigma \rightarrow A_\sigma)$.

(i) Show that $\models M[N/x] \models_\rho = \models M \models_{\rho(x \mapsto \models N \models_\rho)}$

Proof.

M の構造による帰納法で証明

- $M = x$ のとき $\models M[N/x] \models_\rho = \models N \models_\rho = \models M \models_{\rho(x \mapsto \models N \models_\rho)}$
- $M = y \neq x$ のとき、 $\models M[N/x] \models_\rho = \models y \models_\rho = \models M \models_{\rho(x \mapsto \models N \models_\rho)}$
- $M = F(M_1, \dots, M_n)$ のとき、

帰納法の仮定より $\models M_i[N/x] \models_\rho = \models M_i \models_{\rho(x \mapsto \models N \models_\rho)}$

$$\begin{aligned} \text{よって } \models M[N/x] \models_\rho &= \models F(M_1, \dots, M_n)[N/x] \models_\rho \\ &= \models F(M_1[N/x], \dots, M_n[N/x]) \models_\rho \\ &= \models F(\models M_1[N/x] \models_\rho, \dots, \models M_n[N/x] \models_\rho) \\ &= \models F(\models M_1 \models_{\rho(x \mapsto \models N \models_\rho)}, \dots, \models M_n \models_{\rho(x \mapsto \models N \models_\rho)}) \\ &= \models F(M_1, \dots, M_n) \models_{\rho(x \mapsto \models N \models_\rho)} \\ &= \models M \models_{\rho(x \mapsto \models N \models_\rho)} \end{aligned}$$

(ii) Let $(B_\sigma, \models)$ be another Σ -model and $(H_\sigma : A_\sigma \rightarrow B_\sigma)$ be a morphism of Σ -models.

Show that $H(\models M \models_\rho) = \models M \models_{H \circ \rho}$

Proof. M の構造に関する帰納法で証明。 $H(\models x \models_\rho) = H(\rho(x)) = \models x \models_{H \circ \rho}$
 $H(\models F(M_1, \dots, M_n) \models_\rho) = H(\models F(\models M_1 \models_\rho, \dots, \models M_n \models_\rho)) = \models F(M_1, \dots, M_n) \models_{H \circ \rho}$
 $= \models F(H(\models M_1 \models_\rho), \dots, H(\models M_n \models_\rho)) = \models F(\models M_1 \models_{H \circ \rho}, \dots, \models M_n \models_{H \circ \rho}) =$