Exercise 1.6.1

group $|\Sigma| = \{G\}$

$e : \to G$

$\circ : G, G \to G$

$(-)^{-1} : G \to G$

vector space $|\Sigma| = \{K, V\}$

$\cdot : K, V \to V$

$+ : V, V \to V$

Exercise 1.6.2

$$\begin{tikzcd}
\mathrm{FinSign} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Sign} \arrow[d] \\
\mathrm{FinSets} \arrow[r, hook] & \mathrm{Sets}
\end{tikzcd}$$

Exercise 1.6.3

$$
\begin{tikzcd}
\mathrm{Sign}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Sets}^{\to} \arrow[d, "\mathrm{cod}"] \\
\mathrm{Sets} \arrow[r, "T \mapsto T^* \times T"'] & \mathrm{Sets}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Sign} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fam}(\mathrm{Sets}) \arrow[d] \\
\mathrm{Sets} \arrow[r] & \mathrm{Sets}
\end{tikzcd}
$$

(i)

$$|\mathrm{Sign}'| = \{\langle T, O, f\rangle \mid T, O \in \mathrm{Sets},\ f : O \to T^* \times T\}$$

$$\mathrm{Sign}'(\langle T, O, f\rangle, \langle T', O', f'\rangle) = \{\langle h : T \to T',\ h' : O \to O'\rangle \mid (h^* \times h) \circ f = f' \circ h'\}$$

(ii)

$$
\begin{tikzcd}
\mathrm{Fam}(\mathrm{Sets}) \arrow[rr, "\cong"] \arrow[dr] & & \mathrm{Sets}^{\to} \arrow[dl, "\mathrm{cod}"] \\
& \mathrm{Sets} &
\end{tikzcd}
$$

(Proposition 1.2.2) より明らか

(iii) self evident

Exercise 1.6.4 SModel_{ST} を詳しく

$$
\begin{tikzcd}
\mathrm{SModel}_{ST} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{SModel} \arrow[d] \\
\mathrm{Sign}_{ST} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Sign} \arrow[d] \\
\mathrm{FinSets} \arrow[r, hook] & \mathrm{Sets}
\end{tikzcd}
$$

Objects

$(\Sigma, (A_\sigma), [\![-]\!])$ which is a S-Model and $|\Sigma|$ is finite.

Morphisms

S-Model morphism

$(\phi, (H_\sigma)) : (\Sigma, (A_\sigma), [\![-]\!]) \to (\Sigma', (A'_\sigma), [\![-]\!]')$

Exercise 1.6.5

Define a category of signatures of predicates using change-of-base

$$\begin{tikzcd}
\mathbb{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fam}(\mathrm{Sets}) \arrow[d] \\
\mathrm{Sets} \arrow[r, "T \mapsto T^*"'] & \mathrm{Sets}
\end{tikzcd}$$

Define a category of signatures of both function and predicate symbols using change-of-base

$$\begin{tikzcd}
\mathbb{D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fam}(\mathrm{Sets}) \times \mathrm{Fam}(\mathrm{Sets}) \arrow[d] \\
\mathrm{Sets} \arrow[r, "T \mapsto \langle T^* \times T, T^* \rangle"'] & \mathrm{Sets} \times \mathrm{Sets}
\end{tikzcd}$$

Exercise 1.6.6

(i) $X \mapsto \mathrm{Term}(X)$ を $\mathrm{Fam}(\mathrm{Sets})_T \to S\text{-}\mathrm{Model}(\Sigma)$ に拡張せよ

Proof. $f : X \to Y$ in $\mathrm{Fam}(\mathrm{Sets})_T$ が与えられたとする

$\mathrm{Term}(f) : \mathrm{Term}(X) \to \mathrm{Term}(Y)$ in $S\text{-}\mathrm{Model}(\Sigma)$ を

$$\mathrm{Term}(f)(x) = f(x)$$

$$\mathrm{Term}(f)(F(M_1, \cdots, M_n)) = F(\mathrm{Term}(f)(M_1), \cdots, \mathrm{Term}(f)(M_n))$$

で定義する。

$\mathrm{Term}(id) = id$, $\mathrm{Term}(f \circ g) = \mathrm{Term}(f) \circ \mathrm{Term}(g)$ は明らか

(ii) Assume $((A_\sigma)_{\sigma \in T}, [\![\ ]\!])$ is a Σ-model and $(\rho_\sigma : X_\sigma \to A_\sigma)_{\sigma \in T}$.

$([\![-]\!]^\sigma_\rho : \mathrm{Term}(X)_\sigma \to A_\sigma)$ が Σ-models の射であることを示せ。

Proof

$$[\![F(M_1, \cdots, M_n)]\!]_\rho = [\![F]\!]([\![M_1]\!]_\rho, \cdots [\![M_n]\!]_\rho)$$

より

$$\begin{tikzcd}
\mathrm{Term}(X)_{\sigma_1} \times \cdots \times \mathrm{Term}(X)_{\sigma_n} \arrow[r, "{[\![-]\!]^{\sigma_1}_\rho \times \cdots \times [\![-]\!]^{\sigma_n}_\rho}"] \arrow[d, "F"] & A_{\sigma_1} \times \cdots \times A_{\sigma_n} \arrow[d, "{[\![F]\!]}"] \\
\mathrm{Term}(X)_\sigma \arrow[r, "{[\![-]\!]^\sigma_\rho}"'] & A_\sigma
\end{tikzcd}$$

が可換なので。

Exercise 1.6.7

Let $(A_\sigma, [\![-]\!])$ be a Σ-model and ρ a valuation $(X_\sigma \to A_\sigma)$.

(i) Show that $[\![M[N/x]]\!]_\rho = [\![M]\!]_{\rho(x \mapsto [\![N]\!]_\rho)}$

Proof.

Mの構造による帰納法で証明

・$M=x$のとき $[\![M[N/x]]\!]_\rho = [\![N]\!]_\rho = [\![M]\!]_{\rho(x \mapsto [\![N]\!]_\rho)}$

・$M=y \neq x$のとき、$[\![M[N/x]]\!]_\rho = [\![y]\!]_\rho = [\![M]\!]_{\rho(x \mapsto [\![N]\!]_\rho)}$

・$M=F(M_1, \cdots, M_n)$のとき、

帰納法の仮定より $[\![M_i[N/x]]\!]_\rho = [\![M_i]\!]_{\rho(x \mapsto [\![N]\!]_\rho)}$

よって
$$
\begin{aligned}
[\![M[N/x]]\!]_\rho &= [\![F(M_1, \cdots, M_n)[N/x]]\!]_\rho \\
&= [\![F(M_1[N/x], \cdots, M_n[N/x])]\!]_\rho \\
&= [\![F]\!]([\![M_1[N/x]]\!]_\rho, \cdots, [\![M_n[N/x]]\!]_\rho) \\
&= [\![F]\!]([\![M_1]\!]_{\rho(x \mapsto [\![N]\!]_\rho)}, \cdots, [\![M_n]\!]_{\rho(x \mapsto [\![N]\!]_\rho)}) \\
&= [\![F(M_1, \cdots, M_n)]\!]_{\rho(x \mapsto [\![N]\!]_\rho)} \\
&= [\![M]\!]_{\rho(x \mapsto [\![N]\!]_\rho)}
\end{aligned}
$$

(ii) Let $(B_\sigma, [\![-]\!])$ be another Σ-model and $(H_\sigma : A_\sigma \to B_\sigma)$ be a morphism of Σ-models.

Show that $H([\![M]\!]_\rho) = [\![M]\!]_{H \circ \rho}$

Proof. Mの構造に関する帰納法で証明。$H([\![x]\!]_\rho) = H(\rho(x)) = [\![x]\!]_{H \circ \rho}$

$H([\![F(M_1, \cdots, M_n)]\!]_\rho) = H([\![F]\!]([\![M_1]\!]_\rho, \cdots, [\![M_n]\!]_\rho))$
$= [\![F]\!](H([\![M_1]\!]_\rho), \cdots, H([\![M_n]\!]_\rho)) = [\![F]\!]([\![M_1]\!]_{H\circ\rho}, \cdots, [\![M_n]\!]_{H\circ\rho}) = [\![F(M_1, \cdots, M_n)]\!]_{H\circ\rho}$