p.56

$$
\begin{tikzcd}
A \times_{B} E \arrow[r] \arrow[d, "K^*(p)"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
A \arrow[r, "K"'] & B
\end{tikzcd}
$$

$f: X \to Y$ above $Ku: KI \to KJ$ が p-cartesian
$\Rightarrow (u, f): (I, X) \to (J, Y)$ が $K^*(p)$-cartesian

$$
\begin{tikzcd}
(L, Z) \arrow[r, "{(v,h)}"] \arrow[rr, bend left, "{(u \circ v, g)}"] & (I, X) \arrow[r, "{(u,f)}"] & (J, Y)
\end{tikzcd}
$$

$$
\begin{tikzcd}
L \arrow[r, "v"'] & I \arrow[r, "u"'] & J
\end{tikzcd}
$$

f is cartesian over Ku

$$
\begin{tikzcd}
Z \arrow[r, "h"'] \arrow[rr, bend left, "g"] & X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

$$
\begin{tikzcd}
KL \arrow[r, "Kv"'] & KI \arrow[r, "Ku"'] & KJ
\end{tikzcd}
$$

1.5.4 Proposition

$$\mathrm{Sets} \xrightarrow{\coprod} \mathbb{C}$$

$$I \longmapsto I \cdot 1 = \coprod_{i \in I}(1)$$

$$\mathrm{Sets}(I, \mathbb{C}(1, X)) \cong \mathbb{C}\left(\coprod_{i \in I}(1), X\right)$$

P.61 Proposition 1.5.4

$$\pi_X \circ u = \kappa_i \circ !_K$$

$$\begin{tikzcd}
K_j \arrow[r] \arrow[d] \arrow[dd, bend right, "!"'] \arrow[rrdd, bend left=60, "!"] \arrow[dr, phantom, "\lrcorner", very near start] & K \arrow[d, "u"] \arrow[rdd, "!"] & \\
X_j \arrow[r, "\kappa_j"'] \arrow[d, "!_{X_j}"'] & \bigsqcup_{i\in I} X_i \arrow[d, "\pi_X = \bigsqcup_{i\in I} !_{X_i}"] & \\
1 \arrow[r, "\kappa_j"'] & \bigsqcup I & 1 \arrow[l, "\kappa_i"]
\end{tikzcd}$$

$$\begin{tikzcd}
K_j \arrow[d, dashed, "\cong"] \arrow[dr, bend left] \arrow[dd, bend right] & \\
0 \arrow[r] \arrow[d] & 1 \arrow[d, "\kappa_j"] \\
1 \arrow[r, "\kappa_i"'] & \bigsqcup(I)
\end{tikzcd}$$

for $i \neq j$

$$\begin{tikzcd}
K \arrow[d, "\cong"] \arrow[rrd, bend left, "u"] \arrow[rdd, bend right, "!"'] & & \\
K_i \arrow[r, "\kappa_i^*(u)"] & X_i \arrow[r] \arrow[d] & \bigsqcup_{i\in I} X_i \arrow[d, "\pi_X"] \\
& 1 \arrow[r, "\kappa_i"'] & \bigsqcup(I)
\end{tikzcd}$$

よって

$$K \cong \bigsqcup_{j\in I} K_j \cong K_i$$

p.62 Lemma 1.5.5 (ii)

$$\begin{tikzcd} \mathbb{E} \arrow[d, "p"] \\ \mathbb{B} \arrow[d, "r"] \\ \mathbb{A} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{E}_I \arrow[d, "p_I"] \\ \mathbb{B}_I \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{E}_I \arrow[r] \arrow[d, "p_I"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{E} \arrow[d, "p"] \\ \mathbb{B}_I \arrow[r, hook] & \mathbb{B} \end{tikzcd}$$

がPullbackなので change-of-base により p_I は fibration

Exercise 1.5.1

$\{X_i\}_{i \in I}$

FinFam(Sets) なら I が finite

Fam(FinSets) なら各 X_i が finite

Exercise 1.5.2

$$
\begin{tikzcd}
\mathbb{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{UFam}(\omega\text{-Sets}) \arrow[d] \\
\mathrm{PER} \arrow[r, hook] & \omega\text{-Sets}
\end{tikzcd}
$$

Exercise 1.5.3

(i) 証明済み

テキストの $I = pX \in \mathbb{B}$ は間違いだよね？

(ii)
$$\begin{tikzcd} \mathbb{E} \arrow[d, "p"'] \\ \mathbb{B} \arrow[d, "r"'] \\ \mathbb{A} \end{tikzcd}$$

$f : X \to Y \quad I = rpX \in \mathbb{A}$

f is p-cartesian $\Rightarrow f = g \circ h$ with
g is rp-cartesian
h is p_I-caresian

Proof

g を $rp(f)$ の rp-cartesian lifting とする

$$\begin{tikzcd} X \arrow[r, dashed, "h"'] \arrow[rr, bend left, "f"] & X' \arrow[r, "g"] & Y \end{tikzcd}$$

f, g は p-cartesian なので Exercise 1.1.4 (iii) より h も p-cartesian

$$\begin{tikzcd} pX \arrow[r, dashed, "ph"'] \arrow[rr, bend left, "pf"] & pX' \arrow[r, "pg"] & pY \\ I \arrow[r, equal] & I \arrow[r, "rpf"'] & rpY \end{tikzcd}$$

pg は r-cartesian なので左図より

$ph \in \mathbb{B}$

Change-of-base による p_I の定義より h は p_I-cartesian

$f = g \circ h$ with g rp-cartesian, h p_I-cartesian $\Rightarrow$ f is p-cartesian

Proof

above iso ならcartesian

$rp(h) = id_I$ より h は rp-cartesian

cartesian morphism の合成は cartesian なので、$f = g \circ h$ も rp-cartesian

よって f は p-cartesian

Exercise 1.5.4

$$
\begin{tikzcd}
S_c(\mathbb{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Sets}^{\to} \arrow[d, "\mathrm{cod}"] \\
\mathbb{C} \arrow[r, "\Gamma"'] & \mathrm{Sets}
\end{tikzcd}
$$

$\mathbb{C}$がCCC $\Rightarrow$ $S_c(\mathbb{C})$がCCC

Exercise 1.3.7より $\mathrm{Sets}^{\to}$はCCC

Proof

terminal object $\quad (1, \underbrace{\lambda x:1.\, id_1}_{\cong 1 \text{ in } \mathrm{Sets}^{\to}})$

products $\quad (X,\varphi)\times(Y,\psi) \stackrel{\Delta}{=} (X\times Y, \underbrace{\langle -,- \rangle \circ (\varphi\times\psi)}_{\cong \varphi\times\psi \text{ in } \mathrm{Sets}^{\to}})$

exponents $\quad (X,\varphi)\Rightarrow(Y,\psi) \stackrel{\Delta}{=} (X\Rightarrow Y, \theta : U \to \mathbb{C}(1, X\Rightarrow Y))$

where $S = \mathrm{dom}(\varphi)$, $T = \mathrm{dom}(\psi)$

$$
\begin{tikzcd}
U \arrow[r] \arrow[d, "\theta"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \Rightarrow T \arrow[d, "S\Rightarrow\psi"] \\
\mathbb{C}(1, X\Rightarrow Y) \arrow[r] & S \Rightarrow \mathbb{C}(1,Y)
\end{tikzcd}
$$

$$f \longmapsto \lambda s\in S.\ \mathrm{ev}\circ\langle f, \varphi(s)\rangle$$

$$
\begin{aligned}
& S_c(\mathbb{C})((Z, \alpha : W\to\mathbb{C}(1,Z))\times(X, \varphi : S\to\mathbb{C}(1,X)), (Y, \psi : T\to\mathbb{C}(1,Y))) \\
&= \{(j : Z\times X\to Y, i : W\times S\to T) \mid \forall (w,s)\in W\times S.\ \psi(i(w,s)) = j\circ\langle\alpha(w), \varphi(s)\rangle\} \\
&\cong \{(j' : Z\to(X\Rightarrow Y), i' : W\to(S\Rightarrow T)) \mid \forall w\in W, s\in S.\ \psi(i'(w)(s)) = \mathrm{ev}\circ\langle j'\circ\alpha(w), \varphi(s)\rangle\} \\
&= \{\quad " \quad \mid \forall w\in W.\ (S\Rightarrow\psi)(i'(w)) = (\lambda s\in S.\ \mathrm{ev}\circ\langle j'\circ\alpha(w), \varphi(s)\rangle)\} \\
&\cong \{(j' : Z\to(X\Rightarrow Y), k : W\to U) \mid \forall w\in W.\ \theta(k(w)) = j'\circ\alpha(w)\} \\
&= S_c(\mathbb{C})((Z,\alpha), (X,\varphi)\Rightarrow(Y,\psi))
\end{aligned}
$$

$$
\begin{tikzcd}
w \arrow[dd, mapsto] & W \arrow[dr, "k"] \arrow[drr, bend left, "i'"] \arrow[ddr, bend right] & & \\
& & U \arrow[r, hook] \arrow[d, "\theta"'] & S \Rightarrow T \arrow[d, "S \Rightarrow \psi"] \\
j \circ \alpha(w) & & \mathbb{C}(1, X \Rightarrow Y) \arrow[r] & S \Rightarrow \mathbb{C}(1, Y) \\
& & f \arrow[r, mapsto] & \lambda s \in S.\, ev \circ \langle f, \varphi(s) \rangle
\end{tikzcd}
$$

Exercise 1.5.5

Let A be complete lattice

- coproducts $\bigvee_{i\in I} x_i$ in A is universal $\Rightarrow$ A is a frame

proof

$$\begin{tikzcd}
y\wedge x_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & y\wedge\bigvee_{i\in I} x_i \arrow[d] \\
x_i \arrow[r] & \bigvee_{i\in I}
\end{tikzcd}$$

for all $i\in I$

なので universal であることから

$$\bigvee_{i\in I}(y\wedge x_i) = y\wedge\bigvee_{i\in I} x_i$$

よって A は frame

- A is a frame $\Rightarrow$ coproducts $\bigvee_{i\in I} x_i$ are universal

proof

$$\begin{tikzcd}
y_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & z \arrow[d] \\
x_i \arrow[r] & \bigvee_{i\in I} x_i
\end{tikzcd}$$

for all $i\in I$

$y_i = x_i\wedge z$ かつ $z\wedge\bigvee_{i\in I} x_i = z$ より

$$z = z\wedge\bigvee_{i\in I} x_i = \bigvee_{i\in I}(z\wedge x_i) = \bigvee_{i\in I} y_i$$

よって coproducts は universal

coproducts are disjoint

$\Updownarrow$

A has at most 2 elements

とあるが disjoint だと

$$
\begin{tikzcd}
0 \cong X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d] \\
X \arrow[r] & X \vee 1
\end{tikzcd}
$$

となってしまうので 1要素の場合しかダメでは?

Exercise 1.5.6

$\mathbb{B}$ を distributive category とする:

$$(X\times Z)+(Y\times Z) \xrightarrow{[\kappa\times id_Z,\ \kappa'\times id_Z]} (X+Y)\times Z$$

が iso

(i) $X\times X \xrightarrow{\kappa\times id} (X+Y)\times X$ が split mono

Proof

$f=[id,\pi']\circ[\kappa\times id_X,\ \kappa'\times id_X]^{-1}$ とすると

$$\begin{aligned} f\circ(\kappa\times id_X) &= [id,\pi']\circ[\kappa\times id_X,\ \kappa'\times id_X]^{-1}\circ(\kappa\times id_X) \\ &= [id,\pi']\circ[\kappa\times id_X,\ \kappa'\times id_X]^{-1}\circ[\kappa\times id_X,\ \kappa'\times id]\circ\kappa \\ &= [id,\pi']\circ\kappa \\ &= id \end{aligned}$$

(ii) coprojections $X \xrightarrow{\kappa} X+Y \xleftarrow{\kappa} Y$ are mono

Proof

$f,g : Z \rightrightarrows X$ with $\kappa\circ f=\kappa\circ g$ とする

$$\begin{tikzcd}
 & X\times X \arrow[r, "\kappa\times id"] \arrow[d, "\pi"] & (X+Y)\times X \\
Z \arrow[ur, shift left, "\langle f,f\rangle"] \arrow[ur, shift right, "\langle g,f\rangle"'] \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & X &
\end{tikzcd}$$

$\kappa\circ f=\kappa\circ g$

$\Rightarrow (\kappa\times id)\circ\langle f,f\rangle=(\kappa\times id)\circ\langle g,f\rangle$

$\Rightarrow$ — $(\kappa\times id)$ は mono —

$\langle f,f\rangle=\langle g,f\rangle$

$\Rightarrow f=g$

(iii) $0 \to 0\times Z$ が iso

proof

$$
\begin{tikzcd}
0\times Z \arrow[d, "\kappa"'] \arrow[dr, bend left, "id"] & \\
0\times Z + 0\times Z \arrow[r, dashed, "{[id,id]}"] & 0\times Z \\
0\times Z \arrow[u, "\kappa'"] \arrow[ur, bend right, "id"'] &
\end{tikzcd}
$$

左図の $[id,id]$ が iso なので $\kappa = [id,id]^{-1} = \kappa'$

for any $f, g : 0\times Z \rightrightarrows Y$

$$
\begin{tikzcd}
0\times Z \arrow[d, "\kappa"'] \arrow[dr, bend left, "f"] & \\
0\times Z + 0\times Z \arrow[r, dashed, "{[f,g]}"] & Y \\
0\times Z \arrow[u, "\kappa'"] \arrow[ur, bend right, "g"'] &
\end{tikzcd}
$$

$$
\begin{aligned}
f &= [f,g]\circ\kappa \\
&= [f,g]\circ\kappa' \\
&= g
\end{aligned}
$$

よって

$$0\times Z \xrightarrow{\pi} 0 \longrightarrow 0\times Z \;=\; 0\times Z \xrightarrow{id} 0\times Z$$

一方

$$0 \longrightarrow 0\times Z \xrightarrow{\pi} 0 \;=\; 0 \to 0$$ は明らか

ゆえに $0\times Z \xrightarrow{\pi} 0$ は $0 \to 0\times Z$ の inverse

(iv) 0 is strict initial object

すなわち任意の $Z \xrightarrow{f} 0$ はiso

Proof

$$
\begin{tikzcd}
 & 0 \arrow[dd, bend left=60, "!!_Z"] \\
Z \arrow[ur, bend left, "f"] \arrow[r, dashed, "\langle f, id \rangle"] \arrow[dr, bend right, "id"'] & 0 \times Z \arrow[u, "\pi"', "\cong"] \arrow[d, "\pi'"] \\
 & Z
\end{tikzcd}
$$

$$!!_Z \circ f = \pi' \circ \pi^{-1} \circ f = id_Z$$

一方、$f \circ !!_Z = id_0$ は明らか

よって f はiso

Exercise 1.5.7

finite limit は仮定して良いんだよね？

Let $\mathbb{B}$ has universal disjoint coproducts

$$
\begin{tikzcd}
X \arrow[rrd, bend left, "f"] \arrow[rdd, bend right, "g"'] & & \\
& I \arrow[r, "u"] \arrow[d, "\kappa"] & J \arrow[d, "\kappa"] \\
& I+K \arrow[r, "u+v"'] & J+L
\end{tikzcd}
$$

$$\kappa \circ f = (u+v) \circ g$$

coproduct が universal なので以下で定義する $Y \xrightarrow{i} X \xleftarrow{j} Z$ は coproduct.

$$
\begin{tikzcd}
Y \arrow[r, "i"] \arrow[d, "\kappa^*(g)"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "g"] & Z \arrow[l, "j"'] \arrow[d, "\kappa'^*(g)"] \arrow[dl, phantom, "\llcorner", very near start] \\
I \arrow[r, "\kappa"'] \arrow[d, "u"'] & I+K \arrow[d, "u+v"] & K \arrow[l, "\kappa'"] \arrow[d, "v"] \\
J \arrow[r, "\kappa"'] & J+L & L \arrow[l, "\kappa'"]
\end{tikzcd}
$$

一方、coproduct が disjoint なので $Z \to 0$ が存在

$$
\begin{tikzcd}
& & Z \arrow[dll, bend right, "f \circ j"'] \arrow[dl, dashed] \arrow[ddr, bend left, "v \circ \kappa'^*(g)"] & \\
J \arrow[d, "\kappa"'] & 0 \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] & & \\
J+L & L \arrow[l, "\kappa'"] & & L
\end{tikzcd}
$$

$$
\begin{aligned}
\kappa \circ f \circ j &= (u+v) \circ g \circ j \\
&= (u+v) \circ \kappa' \circ \kappa'^*(g) \\
&= \kappa' \circ v \circ \kappa'(g)
\end{aligned}
$$

0 が strict initial なので $Z \cong 0$ で $i : Y \xrightarrow{\cong} X$ なので $h : X \to I$ を $h = \kappa^*(g) \circ i^{-1}$ とおく。$\kappa \circ h = g$ また $\kappa \circ u \circ h = \kappa \circ u \circ \kappa^*(g) \circ i^{-1}$ $= (u+v) \circ \kappa \circ \kappa^*(g) \circ i^{-1} = (u+v) \circ g = \kappa \circ f$ で κ が mono なので $u \circ h = f$

一意性については $X \xrightarrow{h'} I$ が条件を満たすならば $\kappa \circ h' = g = \kappa \circ h$ で、κ が mono なので $h' = h$

$$
\begin{tikzcd}
X \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& I+I \arrow[r, "u+u"] \arrow[d, "\nabla"'] & J+J \arrow[d, "\nabla"] \\
& I \arrow[r, "u"'] & J
\end{tikzcd}
\qquad \nabla\circ f = u\circ g
$$

$$
\begin{tikzcd}
Y \arrow[r, "i"] \arrow[d, "\kappa^*(f)"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] & Z \arrow[l, "j"'] \arrow[d, "\kappa'^*(f)"] \arrow[dl, phantom, "\llcorner", very near start] \\
J \arrow[r, "\kappa"'] & J+J & J \arrow[l, "\kappa'"]
\end{tikzcd}
\qquad Y+Z \xrightarrow[{[i,j]}]{\cong} X
$$

$h=(g\circ i+g\circ j)\circ[i,j]^{-1}:X\to I+I$ とおくと

$$
\begin{aligned}
(u+u)\circ h &= (u+u)\circ(g\circ i+g\circ j)\circ[i,j]^{-1}\\
&= (u\circ g\circ i+u\circ g\circ j)\circ[i,j]^{-1}\\
&= (\nabla\circ f\circ i+\nabla\circ f\circ j)\circ[i,j]^{-1}\\
&= [\kappa\circ\nabla\circ f\circ i,\ \kappa'\circ\nabla\circ f\circ j]\circ[i,j]^{-1}\\
&= [f\circ i,\ f\circ j]\circ[i,j]^{-1}\\
&= f\circ[i,j]\circ[i,j]^{-1}\\
&= f
\end{aligned}
$$

$$
\begin{aligned}
\nabla\circ h &= \nabla\circ(g\circ i+g\circ j)\circ[i,j]^{-1} = [g\circ i,g\circ j]\circ[i,j]^{-1}\\
&= g\circ[i,j]\circ[i,j]^{-1} = g
\end{aligned}
$$

一意性については

$h' : X \to I+I$ が $\nabla \circ h' = g$, $(u+u)\circ h' = f$ を満たすとすると.

$$
\begin{tikzcd}
Y \arrow[r, "i"] \arrow[d, dashed, "h_1"'] \arrow[dd, bend right=60, "\kappa^*(f)"'] & X \arrow[d, "h'"] & Z \arrow[l, "j"'] \arrow[d, dashed, "h_2"] \arrow[dd, bend left=60, "\kappa'^*(f)"] \\
I \arrow[r, "\kappa"] \arrow[d, "u"'] \arrow[dr, phantom, "\lrcorner", very near start] & I+I \arrow[d, "u+u"] & I \arrow[l, "\kappa'"'] \arrow[d, "u"] \arrow[dl, phantom, "\llcorner", very near start] \\
J \arrow[r, "\kappa"'] & J+J & J \arrow[l, "\kappa'"]
\end{tikzcd}
$$

$(u+u)\circ h'\circ i = f\circ i = \kappa \circ \kappa^*(f)$

$(u+u)\circ h'\circ j = f\circ j = \kappa' \circ \kappa'^*(f)$ より $h'_1 : Y \to I$, $h'_2 : Z \to J$ が存在して

$$h' = (h'_1 + h'_2)\circ [i,j]^{-1}$$

ここで

$$h'_1 = [h'_1, h'_2]\circ \kappa = \nabla \circ (h'_1 + h'_2)\circ [i,j]^{-1}\circ i$$
$$= \nabla \circ h' \circ i = g\circ i$$

同様に $h'_2 = g\circ j$

よって $h' = (g\circ i + g\circ j)\circ [i,j]^{-1}$ で一意

すごく冗長なことをしているけど、面倒なので、このままで

coproduct functor $+ : \mathbb{B}\times\mathbb{B}\to\mathbb{B}$ preserves pullbacks.

proof

$$\begin{tikzcd} L \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner" very near start] & K \arrow[d, "v"] \\ I \arrow[r, "u"'] & J \end{tikzcd} \qquad \begin{tikzcd} L' \arrow[r, "q'"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner" very near start] & K' \arrow[d, "v'"] \\ I' \arrow[r, "u'"'] & J' \end{tikzcd}$$ とする

$$\begin{tikzcd} X \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\ & L+L' \arrow[r, "q+q'"] \arrow[d, "p+p'"'] & K+K' \arrow[d, "v+v'"] \\ & I+I' \arrow[r, "u+u'"'] & J+J' \end{tikzcd}$$

$(u+u')\circ f = (v+v')\circ g$ が与えられたとする

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[d, "\kappa^*(f)"'] \arrow[dr, phantom, "\lrcorner" very near start] & X \arrow[d, "f"] & Z \arrow[l, "j"'] \arrow[d, "\kappa'^*(f)"] \arrow[dl, phantom, "\llcorner" very near start] \\ I \arrow[r, "\kappa"'] & I+I' & I' \arrow[l, "\kappa'"] \end{tikzcd}$$

で Y, Z, i, j を定義

$f = (\kappa^*(f)+\kappa'^*(f))\circ[i,j]^{-1}$

$$\begin{tikzcd} Y_1 \arrow[r, "y_1"] \arrow[d, "\kappa^*(g\circ i)"'] \arrow[dr, phantom, "\lrcorner" very near start] & Y \arrow[d, "g\circ i"] & Y_2 \arrow[l, "y_2"'] \arrow[d, "\kappa'^*(g\circ i)"] \arrow[dl, phantom, "\llcorner" very near start] \\ K \arrow[r, "\kappa"'] & K+K' & K' \arrow[l, "\kappa'"] \end{tikzcd} \qquad \begin{tikzcd} Z_1 \arrow[r, "z_1"] \arrow[d, "\kappa^*(g\circ j)"'] & Z \arrow[d, "g\circ j"] & Z_2 \arrow[l, "z_2"'] \arrow[d, "\kappa'^*(g\circ j)"] \\ K \arrow[r, "\kappa"'] & K+K' & K' \arrow[l, "\kappa'"] \end{tikzcd}$$ とすると

$$\begin{tikzcd} Y_2 \arrow[rr, "\kappa'^*(g\circ i)"] \arrow[d, "y_2"'] \arrow[dr] & & K' \arrow[d, "v'"] \\ Y \arrow[d, "\kappa^*(f)"'] & 0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & J' \arrow[d, "\kappa'"] \\ I \arrow[r, "u"'] & J \arrow[r, "\kappa"'] & J+J' \end{tikzcd}$$

$\kappa'\circ v'\circ\kappa'^*(g\circ i) = (v+v')\circ\kappa'\circ\kappa'^*(g\circ i) = (v+v')\circ g\circ i\circ y_2$

$= (u+u')\circ f\circ i\circ y_2 = (u+u')\circ\kappa\circ\kappa^*(f)\circ y_2$

$= \kappa\circ u\circ\kappa^*(f)\circ y_2$ より $Y_2\cong 0$, $Y_1\cong Y$

同様にして $Z_1=0,\ Z_2\cong Z$

$$
\begin{tikzcd}
Y_1 \arrow[dr, dashed, "h_1"] \arrow[drr, bend left, "\kappa^*(g\circ i)"] \arrow[ddr, bend right, "\kappa^*(f)\circ y_1"'] & & \\
& L \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & K \arrow[d, "v"] \\
& I \arrow[r, "u"'] & J
\end{tikzcd}
\qquad
\begin{tikzcd}
Z_2 \arrow[dr, dashed, "h_2"] \arrow[drr, bend left, "\kappa'^*(g\circ j)"] \arrow[ddr, bend right, "\kappa'^*(f)\circ z_2"'] & & \\
& L' \arrow[r, "q'"] \arrow[d, "p'"'] & K' \arrow[d, "v'"] \\
& I' \arrow[r, "u'"'] & J'
\end{tikzcd}
$$

$$\kappa\circ u\circ\kappa^*(f)\circ y_1=(u+u')\circ\kappa\circ\kappa^*(f)\circ y_1=(u+u')\circ f\circ i\circ y_1=(v+v')\circ g\circ i\circ y_1$$
$$=(v+v')\circ\kappa\circ\kappa^*(g\circ i)=\kappa\circ v\circ\kappa^*(g\circ i)$$

κはmonoなので $u\circ\kappa^*(f)\circ y_1=v\circ\kappa^*(g\circ i)$

同様にして、$u'\circ\kappa'^*(f)\circ z_2=v'\circ\kappa'^*(g\circ j)$

$$h=X\xrightarrow{[i,j]^{-1}}Y+Z\xrightarrow{y_1^{-1}+z_2^{-1}}Y_1+Z_2\xrightarrow{h_1+h_2}L+L'$$ とおく

$$(p+p')\circ h=(p\circ h_1\circ y_1^{-1}+p'\circ h_2\circ z_2^{-1})\circ[i,j]^{-1}$$
$$=(\kappa^*(f)+\kappa'^*(f))\circ[i,j]^{-1}=f$$
$$(q+q')\circ h=(q\circ h_1\circ y_1^{-1}+q'\circ h_2+z^{-1})\circ[i,j]^{-1}$$
$$=[\kappa\circ\kappa^*(g\circ i)\circ y_1^{-1},\ \kappa'\circ\kappa'^*(g\circ j)\circ z_2^{-1}]\circ[i,j]^{-1}$$
$$=[g\circ i,\ g\circ j]\circ[i,j]^{-1}=g\circ[i,j]\circ[i,j]^{-1}=g$$

よって hによって $L+L'$は weak pullback

一意性について

$h' : X \to L+L$ が $(p+p')\circ h' = f$, $(q+q')\circ h' = g$ を満たすとする

$$\begin{tikzcd}
Y_1 \arrow[r, "i\circ y_1"] \arrow[d, dashed, "h_1'"] \arrow[dd, bend right=60, "\kappa^*(f)\circ y_1"'] & X \arrow[d, "h'"] \\
L \arrow[r, "\kappa"] \arrow[d, "p"] & L+L' \arrow[d, "p\times p'"] \\
I \arrow[r, "\kappa"] & I+I'
\end{tikzcd}$$

$$\kappa\circ\kappa^*(f)\circ y_1 = f\circ i\circ y_1 = (p+p')\circ h'\circ i\circ y_1$$

$$\begin{tikzcd}
Z_2 \arrow[r, "j\circ z_2"] \arrow[d, dashed, "h_2'"] \arrow[dd, bend right=60, "\kappa'^*(f)\circ z_2"'] & X \arrow[d, "h'"] \\
L' \arrow[r, "\kappa'"] \arrow[d, "p'"] & L+L' \arrow[d, "p+p'"] \\
I' \arrow[r, "\kappa'"] & I+I'
\end{tikzcd}$$

$$\kappa'\circ\kappa'^*(f)\circ z_2 = f\circ j\circ z_2 = (p+p')\circ h'\circ j\circ z_2$$

$$h' = (h_1'+h_2')\circ(y_1^{-1}+z_2^{-1})\circ[i,j]^{-1}$$

$\kappa\circ p\circ h_1' = (p+p')\circ\kappa\circ h_1' = (p+p')\circ h'\circ i\circ y_1 = f\circ i\circ y_1 = \kappa\circ\kappa^*(f)\circ y_1$

κがmonoなので $p\circ h_1' = \kappa^*(f)\circ y_1$

$\kappa\circ q\circ h_1' = (q+q')\circ\kappa\circ h_1' = (q+q')\circ h'\circ i\circ y_1 = g\circ i\circ y_1 = \kappa\circ\kappa^*(g\circ i)$

κがmonoなので $q\circ h_1' = \kappa^*(g\circ i)$

Lがpullbackなので $h_1' = h_1$　同様にして $h_2' = h_2$

よって $h = h'$

補足：universalなcoproductがあれば
distributive categoryになる

$$
\begin{tikzcd}
W \arrow[dr, dashed, "\langle \pi\circ g, f\rangle"] \arrow[drrr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & & \\
& Z\times X \arrow[rr, "\kappa"] \arrow[d, "\pi'"'] \arrow[drr, phantom, "\lrcorner", very near start] & & Z\times(X+Y) \arrow[d, "\pi'"] \\
& X \arrow[rr, "\kappa"'] & & X+Y
\end{tikzcd}
$$

がpullbackでZ×Yも同様

coproductがuniversalなので

$$
\begin{tikzcd}
Z\times X \arrow[r, "id\times\kappa"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z\times(X+Y) \arrow[d, "\pi'"] & Z\times Y \arrow[l, "id\times\kappa'"'] \arrow[d, "\pi'"] \arrow[dl, phantom, "\llcorner", very near start] \\
X \arrow[r, "\kappa"'] & X+Y & Y \arrow[l, "\kappa'"]
\end{tikzcd}
$$

より、 $Z\times X \xrightarrow{id\times\kappa} Z\times(X+Y) \xleftarrow{id\times\kappa'} Z\times Y$ は
coproduct。

よって $[id\times\kappa, id\times\kappa']$ はiso