Exercise 1.1.1 (i)

fとf'が同じmapの上のcartesian morphism
$\mathrm{cod}\, f = \mathrm{cod}\, f'$

$$\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[rr, bend left, "f"] & X' \arrow[r, dashed, "f'"] & Y \\
I \arrow[r, "id"'] & I \arrow[r] & J
\end{tikzcd}$$

$$f = f' \circ \varphi$$

ルーチンワークによりφはiso

Exercise 1.1.1 (ii)

f cartesian

g and h are above same map

$f \circ g = f \circ h$

$$
\begin{tikzcd}
Z \arrow[r, "{g,h}"'] \arrow[rr, bend left, "f \circ g = f \circ h"] & X \arrow[r, dashed, "f"] & Y \\
K \arrow[r, "u"'] & I \arrow[r] & J
\end{tikzcd}
$$

g, h は図を満たす一意な射

Exercise 1.1.2

$p : \mathbb{E} \to \mathbb{B}$

$f : X \to Y$ above u $\quad u = pf$

for each Z, v

$$\mathbb{E}_v(Z, X) \xrightarrow[\Phi]{f \circ (-)} \mathbb{E}_{u \circ v}(Z, Y)$$

Φ iso $\Rightarrow$ f is cartesian

$$\begin{tikzcd}
Z \arrow[r, "\Phi^{-1}(g)"'] \arrow[rr, bend left, "g"] & X \arrow[r, "f"] & Y \\
pZ \arrow[r, "v"'] & pX \arrow[r, "u"'] & pY
\end{tikzcd}$$

一意性は明らか

f is cartesian $\Rightarrow$ Φ iso

$\Psi(f) =$ 一意に決まる射 $Z \to X$ above v とする

一意性から $\Psi\Phi h = h$ for h above v

$\Phi\Psi g = f \circ \Psi g = g$

Exercise 1.1.3 (i)

$$\begin{tikzcd} X \arrow[r, "h"'] \arrow[rr, bend left, "f"] & Z \arrow[r, dashed, "g"'] & Y \end{tikzcd}$$

$$\begin{tikzcd} I \arrow[r, "id"'] & I \arrow[r, "pf"'] & pY \end{tikzcd}$$

pf 上の cartesian morphism を g とおけば
$f = g \circ h$ で h は vertical

Exercise 1.1.3 (ii)

$$\begin{tikzcd} Y \arrow[r, "g"'] \arrow[rr, bend left, "id"] & X \arrow[r, dashed, "f"'] & Y \\ pY \arrow[r, "u^{-1}"'] & pX \arrow[r, "u"'] & pY \end{tikzcd}$$

$f \circ g = id$

$$\begin{tikzcd} X \arrow[r, "id", "g \circ f"'] \arrow[rr, bend left, "f"] & X \arrow[r, dashed, "f"'] & Y \\ pX \arrow[r, "id"'] & pX \arrow[r, "u"'] & pY \end{tikzcd}$$

$$\begin{aligned} p(g \circ f) &= pg \circ pf \\ &= u^{-1} u \\ &= id \end{aligned}$$

$g \circ f = id$

Exercise 1.1.4 (i)

$$
\begin{tikzcd}
Z \arrow[r, "h"'] \arrow[rr, bend left, "g"] & X \arrow[r, "f", "\cong"'] & Y \\
pZ \arrow[r, "v"'] & pX \arrow[r, "u"'] & pY
\end{tikzcd}
$$

$pf = u$

$pg = u \circ v$

$ph = v$

$g = f \circ h$

$h = f^{-1} \circ g$ は条件を満たす一意な射

Exercise 1.1.4 (ii)

$f: X \to Y$ is cartesian

$g: Y \to Z$ is cartesian

$$\begin{tikzcd}
W \arrow[r, "j"'] \arrow[rr, bend left, "i"] \arrow[rrr, bend left=50, "h"] & X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \\
pW \arrow[r] & pX \arrow[r] & pY \arrow[r] & pZ
\end{tikzcd}$$

gがcartesianであることからiが一意に決まる

fがcartesianであることからiに対してjが一意に決まる

Exercise 1.1.4 (iii)

$f : X \to Y$
$g : Y \to Z$ is cartesian
$g \circ f : X \to Z$ is cartesian

$$\begin{tikzcd} W \arrow[r, "i"'] \arrow[rr, bend left, "h"] \arrow[rrr, bend left=50, "g \circ h"] & X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

$$\begin{tikzcd} pW \arrow[r, "w"'] & pX \arrow[r] & pY \arrow[r] & pZ \end{tikzcd}$$

$g \circ f$ がcartesianなので $g \circ h$ に対して
i が一意に決まる

g がcartesianなので $h = f \circ i$

$g \circ h = g \circ f \circ i$ となる $i : W \to X$ above w
が一意に決まる

$g \circ h = g \circ f \circ i = g \circ f \circ i' \Rightarrow i = i'$

$h = f \circ i = f \circ i' \Rightarrow i = i'$

$h = f \circ i$ となる $i : W \to X$ above w も一意

Exercise 1.1.4 (iv)

$$\begin{tikzcd}
\mathbb{E} \arrow[d, "p"] & Z \arrow[r, "h"] \arrow[rr, bend left, "g \in Cart(\mathbb{E})(Z,Y)"] & X \arrow[r, "f"] & Y \\
\mathbb{B} & pZ \arrow[r] & I \arrow[r, "u"'] & pY
\end{tikzcd}$$

pがcartesianなので、u上のcartesian morphism fが存在

gに対してhが一意に存在

$g = f \circ h$とfがcartesianなので、(iii)より

hもcartesian

よって、$|p| : Cart(\mathbb{E}) \to \mathbb{B}$はfibration

Exercise 1.1.3 (ii)より、verticalなcartesian morphismはisoなので、$|p|$のfibreの射はすべてisoで、$|p|$のfibreはgroupoid.

Exercise 1.1.5

$$
\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "(\mathrm{B})"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "(\mathrm{A})"] & \cdot \arrow[d] \\
\cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}
$$

$\begin{array}{c} \mathbb{B}^{\to} \\ \mathrm{cod} \downarrow \\ \mathbb{B} \end{array}$ に対して 1.1.4 (ii), (iii) を適用

Exercise 1.1.6 if part

$$\begin{tikzcd}
Z \arrow[r, "i"] \arrow[rrr, bend left, "g"] & W \arrow[r, "h"] & X \arrow[r, "f"] & Y \\
pZ \arrow[r, "v"'] & I \arrow[rr, "u"'] & & pY
\end{tikzcd}$$

uのweak cartesian morphismをfとする

g above $u\circ v$が与えられたとする

vのweak cartesian liftingをhとする

(b)より$f\circ h$はweak cartesian

vertian map i で $g=f\circ h\circ i$ を満たすものが一意に存在

$j:Z\to X$ above v が $g=f\circ j$ を満たしたとする

hがweak cartesianなので $j=h\circ i'$ を満たすvertical map i' が一意に存在

$g=f\circ j=f\circ h\circ i'$

上のiの一意性より $i'=i$ で $j=h\circ i$ は $g=f\circ j$ を満たす一意な射

よってfはcartesian

Exercise 1.1.6 only if part

(a) cartesian ならば weak cartesian なので
自明

(b) $f: X \to Y$ が weak cartesian だとすると
$pf: pX \to pY$ 上の cartesian morphism
$f': X' \to Y$ が存在

f, f' は共に pf 上 weak cartesian
なので E/Y で同型

weak cartesian は cartesian と一致

cartesian morphism は合成に
ついて閉じている

Exercise 1.1.7

$$\begin{tikzcd} \mathbb{B} \times \mathbb{C} \arrow[d, "fst"'] \\ \mathbb{B} \end{tikzcd} \qquad \begin{tikzcd} (K, Z) \arrow[r, "{(v, f)}"'] \arrow[rr, bend left, "{(u \circ v, f)}"] & (I, Y) \arrow[r, "{(u, id)}"'] & (J, Y) \\ K \arrow[r, "v"'] & I \arrow[r, "u"'] & J \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{B} \arrow[d, "id"'] \\ \mathbb{B} \end{tikzcd} \qquad \begin{tikzcd} K \arrow[r, "v"'] \arrow[rr, bend left, "u \circ v"] & I \arrow[r, "u"'] & J \\ K \arrow[r, "v"'] & I \arrow[r, "u"'] & J \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{B} \arrow[d] \\ 1 = \{*\} \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"'] \arrow[rr, bend left, "f"] & Y \arrow[r, "id"'] & Y \\ 1 \arrow[r] & 1 \arrow[r] & 1 \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[d] \\ I \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "="'] \arrow[rr, bend left, "="] & x \arrow[r, "="'] & x \\ i \arrow[r, "="'] & i \arrow[r, "="'] & i \end{tikzcd}$$

Exercise 1.1.8　$\mathrm{dom} : \mathbb{B}^{\to} \to \mathbb{B}$

(i) opslice category $I \backslash \mathbb{B}$

object $I \to J$

morphism
$$
\begin{tikzcd}
 & I \arrow[dl] \arrow[dr] & \\
J \arrow[rr, "f"'] & & K
\end{tikzcd}
$$

(ii)
$$
\begin{tikzcd}
K \arrow[r, "v"] \arrow[d, "\psi"'] & I \arrow[r, "u"] \arrow[d, "\varphi \circ u"] & J \arrow[d, "\varphi"] \\
Z \arrow[r, "f"] \arrow[rr, bend right, "f"'] & Y \arrow[r, "id"'] & Y
\end{tikzcd}
$$

外側が可換なので

$\varphi \circ u \circ v = f \circ \psi$

よって左側の四角形も可換

一意性は明らか

(iii) $\mathrm{dom} : \mathbb{B}/I \to \mathbb{B}$

$$
\begin{tikzcd}
Z \arrow[r, "v"'] \arrow[rr, bend left, "u \circ v"] \arrow[drr, bend right, "\psi"'] & X \arrow[r, "u"] \arrow[dr, "\varphi \circ u"'] & Y \arrow[d, "\varphi"] \\
 & & I
\end{tikzcd}
$$

外側が可換なので

$\varphi \circ u \circ v = \psi$

よって左側の三角形も可換

一意性は明らか

$$
Z \xrightarrow{v} X \xrightarrow{u} Y
$$

Exercise 1.1.9

$$\begin{array}{ccc} \mathbb{B}^{\to\to} & & \xrightarrow{f}\xrightarrow{g} \\ \downarrow & & \downarrow \\ \mathbb{B}^{\to} & & \xrightarrow{g} \end{array}$$

$$\begin{tikzcd}
X'' \arrow[r] \arrow[d] \arrow[rr, bend left] & X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\
Y'' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Y \arrow[d] \\
Z'' \arrow[r] & Z' \arrow[r] & Z
\end{tikzcd}$$

X′がpullbackなので
条件を満たす$X'' \to X$は
一意に存在

∴ $\mathbb{B}^{\to\to} \to \mathbb{B}^{\to}$はfibration

$$\begin{array}{ccc} \mathbb{B}^{\to\to} & & X \xrightarrow{f} Y \xrightarrow{g} Z \\ \downarrow & & \downarrow \\ \mathbb{B}^{\to} & & Y \xrightarrow{g} Z \\ \downarrow & & \downarrow \\ \mathbb{B} & & Z \end{array}$$

$$\begin{tikzcd}
X'' \arrow[r] \arrow[d] \arrow[rr, bend left] & X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\
Y'' \arrow[r] \arrow[d] \arrow[rr, bend left] & Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d] \\
Z'' \arrow[r] & Z' \arrow[r] & Z
\end{tikzcd}$$

Y′がpullbackなので条件を満たす
$Y'' \to Y'$は一意
X′がpullbackなので条件を
満たす$X'' \to X'$は一意
∴青字をcartesian liftingとしてfibration

Exercise 1.1.10 $\mathbf{Cat} \to \mathbf{Sets}$

$$
\begin{tikzcd}
\mathbf{Cat} \arrow[d, "\mathrm{Ob}"] & \mathbb{E} \arrow[r, "H"] \arrow[rr, bend left, "G"] & \mathbb{D} \arrow[r, "F"] & \mathbb{C} \\
\mathbf{Sets} & \mathrm{Ob}\,\mathbb{E} \arrow[r, "v"'] & S \arrow[r, "u"'] & \mathrm{Ob}\,\mathbb{C}
\end{tikzcd}
$$

$\mathrm{Ob}\,\mathbb{D} = S$

$\mathbb{D}(X, Y) = \mathbb{C}(uX, uY)$

$FX = uX$

$Ff = f$

$HX = vX$

$Hf = Gf$

$FHX = uvX = GX$

$FHf = Gf$

$\therefore FH = G$

v上なので object part は一意

Fが faithful なので morphism part も一意

$\therefore$ Hは一意

$\therefore$ F は u の cartesian lifting で

Ob: Cat→Sets は fibration

Exercise 1.1.10 $Sp \to Sets$

$$
\begin{tikzcd}
Sp \arrow[d, "|\cdot|"'] \\
Sets
\end{tikzcd}
\qquad
\begin{tikzcd}
Z \arrow[r, "h"] \arrow[rr, bend left, "g"] & X \arrow[r, "f"] & Y \\
|Z| \arrow[r, "v"'] & I \arrow[r, "u"'] & |Y|
\end{tikzcd}
$$

$f(x) = u(x)$

$\Omega X = \{ f^{-1}O \mid O \in \Omega Y \}$

$f^{-1}O \in \Omega X$ for all $O \in \Omega Y$ なので f は連続

$h(z) = v(z)$

$h^{-1}f^{-1}O = g^{-1}O \in \Omega Z$ なので h は連続

v 上なので h は一意

$\therefore$ f は u の cartesian lifting

$\therefore$ $|\cdot| : Sp \to Sets$ は fibration

Exercise 1.1.11

$$
\begin{tikzcd}
\mathrm{Vect} \arrow[d, "|\cdot|"] & (M, V, \circ) \arrow[r, "{(v,f)}"] \arrow[rr, bend left, "{(u \circ v, f)}"] & (K, W, \bullet) \arrow[r, "{(u, \mathrm{id})}"] & (L, W, \cdot) \\
\mathrm{Fld} & M \arrow[r, "v"'] & K \arrow[r, "u"'] & L
\end{tikzcd}
$$

$$a \bullet x \doteq u(a) \cdot x$$

$$\mathrm{id}(a \bullet x) = u(a) \cdot x = u(a) \cdot \mathrm{id}(x)$$

(v, f)が条件を満たす一意な射であることは明らか

fibre Vect_K は K上のベクトル空間の圏？

cartesian morphism は (u, f) where f は iso